Hyperbolic Conservation Laws

Rinaldo M. Colombo, M. Garavello, G. Guerra

Milano–Bicocca University

Milano, 09.01.2013

Outline

Dates

Features

Results and Problems

Hyperbolic Conservation (Balance) Laws

$$\partial_t u + \operatorname{div}_x f(t,x,u) = g(t,x,u) \qquad \begin{array}{r} t \in \mathbb{R}^+ \\ x \in \mathbb{R}^N \\ u \in \mathbb{R}^n \end{array}$$

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Euler 1755

Hyperbolic Conservation (Balance) Laws

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Euler 1755

Existence
Dependence from Data
Dependence from f, g

Hyperbolic Conservation (Balance) Laws

$$\partial_t u + \mathrm{div}_x f(t,x,u) = g(t,x,u) \qquad \begin{array}{r} t \in \mathbb{R}^+ \\ x \in \mathbb{R}^N \\ u \in \mathbb{R}^n \end{array}$$

Euler 1755

TWO theories

Existence

Dependence from Data

Dependence from f, g

Hyperbolic Conservation (Balance) Laws

$$\partial_t u + \mathrm{div}_x f(t,x,u) = g(t,x,u) \qquad \begin{array}{r} t \in \mathbb{R}^+ \\ x \in \mathbb{R}^N \\ u \in \mathbb{R}^n \end{array}$$

Euler 1755

$N = 1$ $\quad n \geq 1$ TWO theories $N \geq 1$ $\quad n = 1$

$$\partial_t u + \partial_x f(u) = 0 \qquad\qquad \partial_t u + \mathrm{div}_x f(t,x,u) = g$$

Existence

Dependence from Data

Dependence from f, g

Hyperbolic Conservation (Balance) Laws

$$\partial_t u + \operatorname{div}_x f(t,x,u) = g(t,x,u) \qquad \begin{array}{r} t \in \mathbb{R}^+ \\ x \in \mathbb{R}^N \\ u \in \mathbb{R}^n \end{array}$$

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1965

Existence

Dependence from Data

Dependence from f, g

(Glimm: CPAM, 1965)

Hyperbolic Conservation (Balance) Laws

$$\partial_t u + \operatorname{div}_x f(t,x,u) = g(t,x,u) \qquad \begin{array}{l} t \in \mathbb{R}^+ \\ x \in \mathbb{R}^N \\ u \in \mathbb{R}^n \end{array}$$

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1965 Existence

1995 Dependence from Data

Dependence from f, g

(Bressan & Colombo: ARMA, 1995)

(Bressan, Crasta & Piccoli: Mem. AMS, 2000)

Hyperbolic Conservation (Balance) Laws

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1965	Existence
1995	Dependence from Data
2002	Dependence from f, g

(Bianchini & Colombo: PAMS, 2002)

Hyperbolic Conservation (Balance) Laws

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Euler 1755

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$\partial_t u + \partial_x f(u) = 0$		$\partial_t u + \operatorname{div}_x f(t,x,u) = g$
1965	Existence	1970
1995	Dependence from Data	
2002	Dependence from f, g	

(Kružkov: Mat. Sb., 1970)

Hyperbolic Conservation (Balance) Laws

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1995	Dependence from Data	1970
2002	Dependence from f, g	2009

(Colombo, Mercier & Rosini: Comm. Math. Sci., 2009)

Hyperbolic Conservation Laws

Outline

Dates

Features

Results and Problems

Hyperbolic Conservation Laws – Features

$$\partial_t u + \operatorname{div}_x f(t, x, u) = g(t, x, u)$$

x	$\in$	$\mathbb{R}^N$	space
t	$\in$	$\mathbb{R}^+$	time
u	$\in$	$\mathbb{R}^n$	unknown
f	is	smooth	flow $(n \times N)$
g	is	smooth	o.d.e.

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1. Evolution equations

Hyperbolic Conservation Laws – Features

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1. Evolution equations
2. Singularities arise

Hyperbolic Conservation Laws – Features

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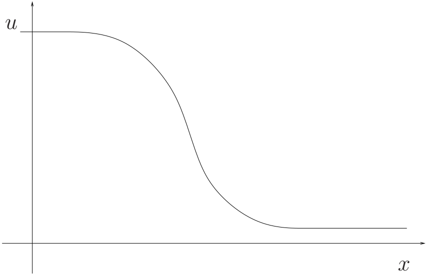
1. Evolution equations
2. Singularities arise
   $\partial_t u + \lambda\, \partial_x u = 0$



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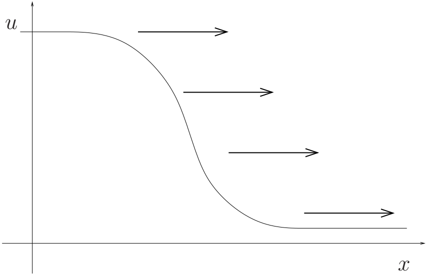
   $u(t,x) = u_o(x - \lambda \, t)$



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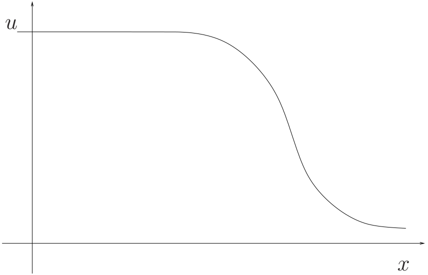

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1. Evolution equations
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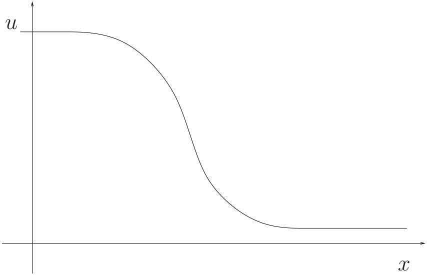
   $\partial_t u + f'(u)\, \partial_x u = 0$



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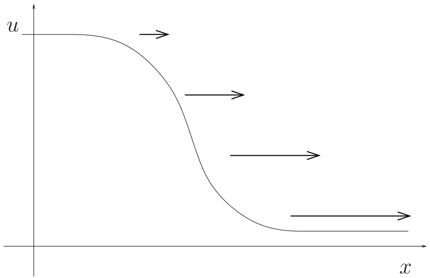

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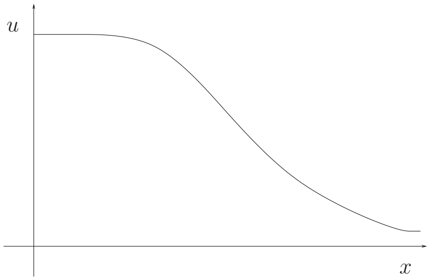

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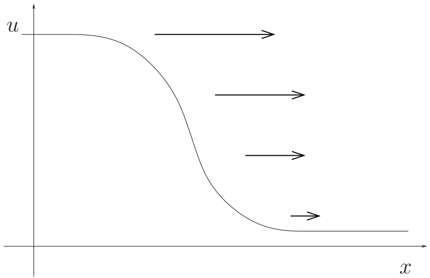

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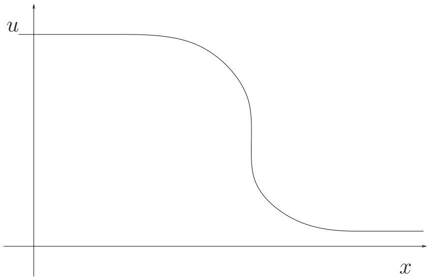

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1. Evolution equations
2. Singularities arise
3. Non reversible (entropy!)

Hyperbolic Conservation Laws – Features

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1. Evolution equations
2. Singularities arise
3. Non reversible
4. Finite Propagation Speed

Hyperbolic Conservation Laws – Features

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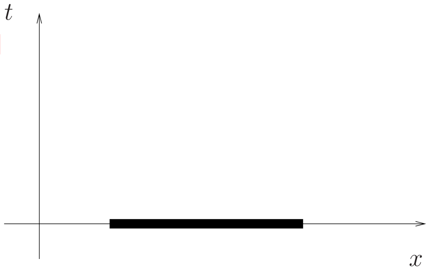

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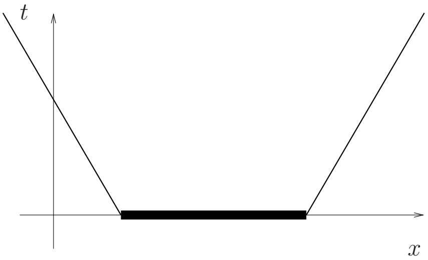

Hyperbolic Conservation Laws – Features

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1. Evolution equations
2. Singularities arise
3. Non reversible
4. Finite Propagation Speed
5. $g = 0$ Conservation

$$\int_\Omega \left[u(t_2,x) - u(t_1,x)\right] \mathrm{d}x = \int_{t_1}^{t_2} \int_{\partial\Omega} f\left(t,x,u(t,x)\right) \cdot \nu(x) \,\mathrm{d}x\,\mathrm{d}t$$

Hyperbolic Conservation Laws – Features

$$\partial_t u + \mathrm{div}_x f(t,x,u) = g(t,x,u)$$

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1. Evolution equations
2. Singularities arise
3. Non reversible
4. Finite Propagation Speed
5. $g = 0$ Conservation
6. Balance

$$\int_\Omega \left[u(t_2,x) - u(t_1,x)\right] \mathrm{d}x = \int_{t_1}^{t_2} \int_{\partial\Omega} f\left(t,x,u(t,x)\right) \cdot \nu(x) \,\mathrm{d}x\,\mathrm{d}t + \int_{t_1}^{t_2} \int_\Omega g\left(t,x,u(t,x)\right) \mathrm{d}t\,\mathrm{d}x$$

Hyperbolic Conservation Laws

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Hyperbolic Conservation Laws

Pure Analysis

Hyperbolic Conservation Laws

Pure Analysis

The case $n \geq 1$, $N \geq 1$

Hyperbolic Conservation Laws

Pure Analysis

Large Total Variation

(Bianchini, Colombo & Monti: JDE, 2010)

(Colombo & Monti: CPAA, 2010)

Hyperbolic Conservation Laws

Pure Analysis

Stability Estimates

(Colombo, Mercier & Rosini: Comptes Rendus Math., 2009)

(Colombo, Mercier & Rosini: Comm. Math. Sc., 2009)

(Mercier: JHDE, 2012)

Gas Dynamics

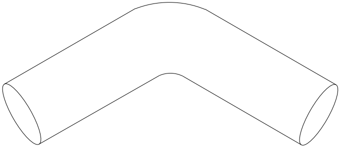
Gas Dynamics

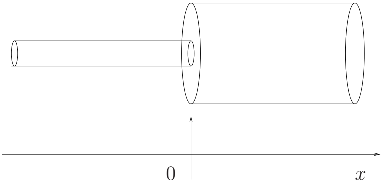
VACUUM

Hyperbolic Conservation Laws

Gas Dynamics

Junctions and Pipelines





(Colombo & Marcellini: NHM, 2010)

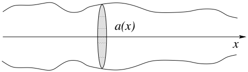
(Colombo & Marcellini: JMAA, 2010)

(Colombo & Garavello: SIAM J. Math. An., 2008)

Hyperbolic Conservation Laws

Gas Dynamics

Gas flow in a tube with variable section



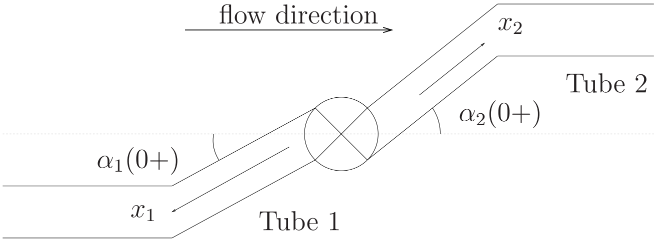
$$\begin{cases} \partial_t \rho + \partial_x q = -\frac{a'(x)}{a(x)}\, q \\ \partial_t q + \partial_x \left(\frac{q^2}{\rho} + p\right) = -\frac{a'(x)}{a(x)}\, \frac{q^2}{\rho} \\ \partial_t e + \partial_x \left(\frac{q}{\rho}\, (e+p)\right) = -\frac{a'(x)}{a(x)} \left(\frac{q}{\rho}(e+p)\right). \end{cases}$$

(Guerra, Marcellini & Sachers: SIAM J. Math. An., 2009)

Hyperbolic Conservation Laws

Gas Dynamics

Control



(Colombo, Guerra, Herty & Sachers: SIAM J. Control Opt., 2009)

(Colombo, Herty & Sachers: SIAM J. Math. An., 2008)

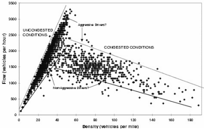
Hyperbolic Conservation Laws

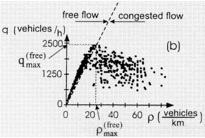
Vehicular Traffic

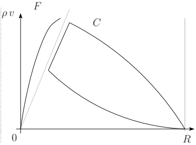
Hyperbolic Conservation Laws

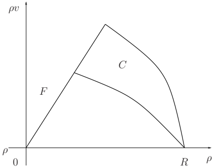
Vehicular Traffic

Models









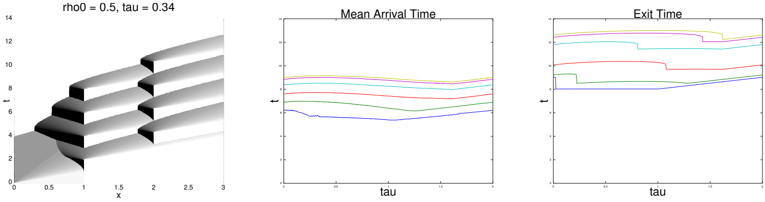
(Colombo, Marcellini & Rascle: SIAM J. Appl. Math., 2010)

(Mercier: JMAA, 2009)

Hyperbolic Conservation Laws

Vehicular Traffic

Control

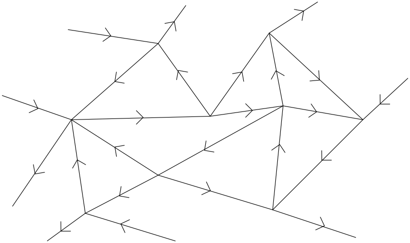


(Colombo, Goatin & Rosini: ESAIM M2AN, 2011)

Hyperbolic Conservation Laws

Vehicular Traffic

Junctions & Networks



(Garavello & Goatin: DCDS A, 2012)

(Garavello & Piccoli: Ann. IHP, 2009)

(Garavello & Piccoli: Book, 2006)

(Coclite, Garavello & Piccoli: SIAM J. Math. An, 2005)

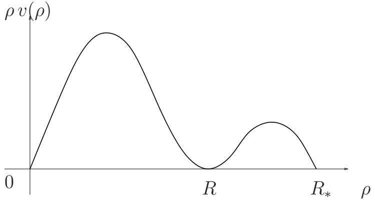
Hyperbolic Conservation Laws

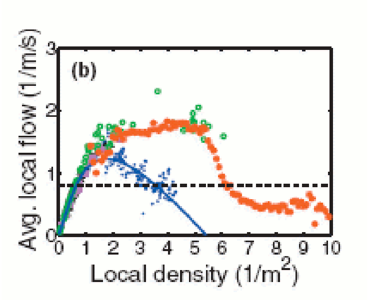
Crowd Dynamics

Hyperbolic Conservation Laws

Crowd Dynamics

Models





(Colombo & Rosini: M2AS, 2005)

(Helbing et al: Phys.Review E, 2007)

(Colombo, Garavello, Mercier: M3AS, 2012)

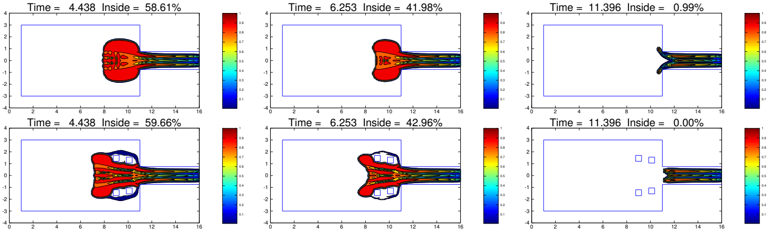
(Colombo, Garavello, Mercier: Comptes Rendus Math., 2011)

(Colombo & Mercier: ActaMath. Sc., 2011)

Hyperbolic Conservation Laws

Crowd Dynamics

Control



(Colombo, Herty & Mercier: ESAIM COCV, 2011)

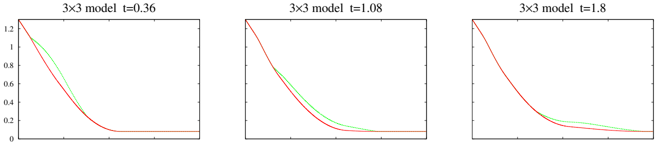
Hyperbolic Conservation Laws

Granular Materials

Hyperbolic Conservation Laws

Granular Materials

Models



$$\begin{cases} \partial_t h + \partial_x (hv) = -\gamma \left(\alpha - |\partial_x u|\right) h \\ \partial_t (hv) + \partial_x (hv^2 + \frac{1}{2} g\, h^2) = -g\, h\, \partial_x u - \gamma \left(\alpha - |\partial_x u|\right) h\, v + fh \\ \partial_t u = \gamma \left(\alpha - |\partial_x u|\right) h \end{cases}$$

(Colombo, Guerra, Monti: IMA J. Appl. Math, 2012)

(Cattani, Colombo, Guerra: ZAMM, 2011)

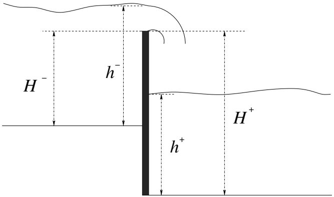
Hyperbolic Conservation Laws

Water Distribution

Hyperbolic Conservation Laws

Water Distribution

Models



$$\begin{cases} \partial_t h + \partial_x (hv) = 0 \\ \partial_t (hv) + \partial_x \left(hv^2 + \frac{1}{2} g h^2 \right) = 0 \end{cases}$$

$$\begin{cases} h^- v^- = C \left(\left[h^- - H^- \right]_+ - \left[h^+ - H^+ \right]_+ \right) \cdot \sqrt{\left| [h^- - H^-]_+ - [h^+ - H^+]_+ \right|} \\ h^+ v^+ = h^- v^-, \end{cases}$$

(Guerra, Herty & Marcellini: NHM, 2011)

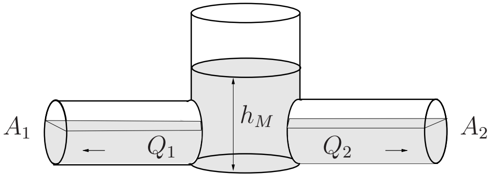
Hyperbolic Conservation Laws

HCL + ODEs

Hyperbolic Conservation Laws

HCL + ODEs

Sewer Systems – Blood Circulation



(Borsche, Colombo & Garavello: Preprint, 2012)

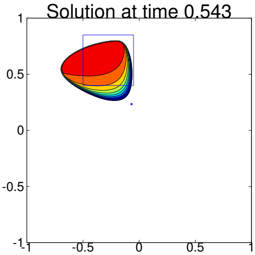
(Borsche, Colombo & Garavello: JDE, 2012)

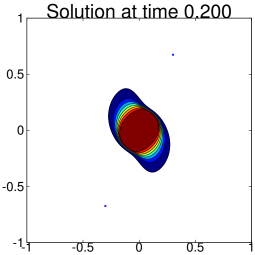
(Borsche, Colombo & Garavello: Nonlinearity, 2010)

Hyperbolic Conservation Laws

HCL + ODEs

Individuals–Populations Interactions



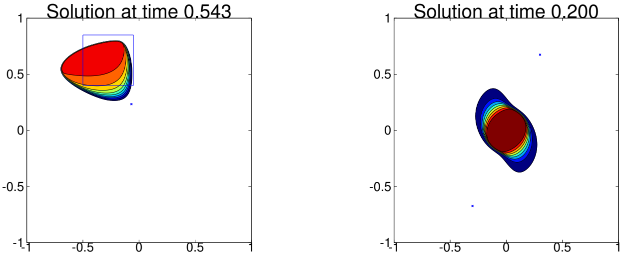


(Colombo & Mercier: J. Nonlinear Sc., 2012)

Hyperbolic Conservation Laws

HCL + ODEs

Individuals–Populations Interactions



Bridges!

(Colombo & Mercier: J. Nonlinear Sc., 2012)

Hyperbolic Conservation Laws

Rinaldo M. Colombo, M. Garavello, G. Guerra

Milano–Bicocca University

Milano, 09.01.2013