MODELLING SUPPLY NETWORKS WITH PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. A continuum-discrete model for supply networks is introduced. The model consists of a system of conservation laws: a conservation law for the goods density and an evolution equation for the processing rate. The network is formed by sub-chains and nodes at which, motivated by real cases, two routing algorithms are considered: the first maximizes fluxes taking into account goods' final destinations, while the second maximizes fluxes without constraints. We analyze waves produced at nodes and equilibria for both algorithms, relating the latter to production rates in real supply networks. In particular, we show how the model can reproduce the well-known Bullwhip effect.

1. Introduction.

A supply network can be considered as an organization of activities, that performs the functions of materials procurement, their transformation into intermediate and finished goods, and the distribution of these finished products to customers. It is evident that the term supply network can be seen in a very general way, since it is about the goods production and their distribution to the final user.

In last years, the interest of scientific community for supply chains and networks modelling has become greater and greater. The main aim is to plan supply networks in such way to reduce the dead times and to avoid bottlenecks, obtaining as a result a greater coordination leading to the optimization of the production process of a given good. Supply networks modelling is characterized by different mathematical approaches: on the one hand, there are discrete event simulations based on considerations of individual parts. On the other, continuous models (for a general overview see [1], [2], [3], [4]), based on partial differential equations, have been introduced. Probably the first paper for supply chains in this direction was [2] where the authors, taking the limit on the number of parts and suppliers, have obtained a conservation law, whose flux is described by the minimum among the parts density and the maximal productive capacity.
Due to the difficulty of finding solution for the general equation proposed in [2], other fluid dynamic models for supply chains were introduced in [9], [6] and [13]. The work [9] is based on a mixed continuum-discrete model, i.e. the supply chain is described by continuous arcs and discrete nodes, it means that the load dynamics is solved in a continuous way on the arcs, and at the nodes imposing the conservation of the goods density, but not of the processing rate. In fact, each arch is modelled by a system of two equations: a conservation law for the goods density, and

Date: October 1, 2007 and, in revised form, , .
2000 *Mathematics Subject Classification.* Primary 8F15, 58F17; Secondary 53C35.
Key words and phrases. Conservation laws, supply networks.

an evolution equation for the productive capacity. Possible choices of solutions at nodes guaranteeing the conservation of fluxes are discussed, and a Riemann Solver is defined, fixing the rule:

SC1 The incoming density flux is equal to the outgoing density flux. Then, if a solution with only waves in the density exists, then such solution is taken, otherwise the minimal processing rate wave is produced.

Moreover existence of solutions to Cauchy problems was proved. The paper [13] considers a conservation law, with constant processing rate, inside each supply sub-chain, with an entering queue for exceeding parts. The dynamics at a node is solved considering an ode for the queue. Some optimization technique for the model described in [13] is developed in [14], while the existence of solutions to Cauchy problems with the front tracking method is proved in [15]. It is evident that the models described in [9] and [13] complete each other. In fact, the approach of [13] is more suitable when the presence of queue with buffer is fundamental to manage goods production. The model of [9], on the other hand, is useful when there is the possibility to reorganize the supply chain: particularly, the productive capacity can be readapted for some contingent necessity.
Starting from the model introduced in [9] and fixing the rule that the objects are processed in order to maximize the flux, two different Riemann Solvers are described and equilibria at a node are discussed in [6]. Moreover, discretization algorithms to find approximated solution to the problem are described, numerical experiments on sample supply chains are reported and discussed for both the Riemann Solvers. While the papers [2], [9], [6] treat the case of chains, i.e. sequential processors, modelled by a real line seen as a sequence of sub-chains corresponding to real intervals, the model in [13] and the extended results in [14], [15] refer to networks.

The aim of this paper is to extend the continuum-discrete model of [9] and [6], regarding sequential supply chains, to supply networks which consist of sub-chains and two types of nodes: nodes with one incoming sub-chain and more outgoing ones and nodes with more incoming sub-chains and one outgoing sub-chain. In fact, these two types of nodes are the most common in real supply networks.

Definition 1.1. (Network definition) A supply network is a finite, connected directed graph consisting of a finite set of arcs (sub-chains) $\mathcal{I} = \{I_k : k = 1, ..., N+1\}$ and a finite set of junctions $\mathcal{P}$.

On each sub-chain the load dynamic is given by a continuum system of type

$$\rho_t + f_\varepsilon^k(\rho, \mu)_x = 0, \tag{1.1}$$

$$\mu_t - \mu_x = 0, \tag{1.2}$$

where ρ is the density of objects processed by the supply chain network, μ is the processing rate and f_ε^k is the flux given by

$$f_\varepsilon^k(\rho, \mu) = \begin{cases} \rho, & 0 \le \rho \le \mu, \\ \mu + \varepsilon(\rho - \mu), & \mu \le \rho \le \rho_k^{\max}, \end{cases}$$

with $\rho_k^{\max}$ and $\mu_k^{\max}$ the maximum density and processing rate on the sub-chain I_k. We interpret the evolution at a node P thinking to it as a Riemann Problem (RP) for the density equation (1.1) with μ data as parameters. The Riemann Problems are solved fixing two "routing" algorithms:

RA1 Goods from an incoming sub-chain are sent to outgoing ones according to their final destination in order to maximize the flux over incoming sub-chains. Goods are processed ordered by arrival time (FIFO policy).

RA2 Goods are processed by arrival time (FIFO policy) and are sent to outgoing sub-chains in order to maximize the flux over incoming and outgoing sub-chains.

The two algorithms were already used in [10] for the analysis of packets flows in telecommunication networks. Notice that the second algorithm allows the redirection of goods, taking into account possible high loads of outgoing sub-chains. For both routing algorithms the flux of goods is maximized considering one of the two additional rules, SC2 and SC3 (see [9]):

SC2 The objects are processed in order to maximize the flux with the minimal value of the processing rate.

SC3 The objects are processed in order to maximize the flux. If a solution with only waves in the density ρ exists, then such solution is taken, otherwise the minimal μ wave is produced.

The rules SC2 and SC3 seem to be more *elastic* than SC1, allowing more rich dynamics. According to these routing algorithms we define Riemann Solvers and discuss the waves formation at nodes. The detailed analysis of waves permits to better visualize and understand the dynamics effects on sub-chains of the defined Riemann Solvers. We provide explicit examples to illustrate the differences between RA1 and RA2 and also between SC2 and SC3.

Then we consider generic equilibria with active and not active constraints for the maximization problem solved at nodes. We relate the found results with the counterpart in real supply networks. Then we show that SC3 reproduces the well known Bullwhip effect (see [8], [11], [12], [17], [18], [20]), i.e., under certain conditions (delays in adaptation of production or delivery rates), the oscillations in delivery and in the resulting inventories (stock level of the products) grow from one producer to the next upstream one, leading to instability respect to perturbation in the production rate. The latter confirms that SC3 appears to be the more realistic modelling choice.

The outline of the paper is the following. In Section 2 some examples of real supply networks are introduced in order to motivate the rules introduced to solve the dynamics at nodes. Section 3 gives the basic definitions of supply network and Riemann Solver. In Section 4, Riemann Solvers at junctions are defined for both algorithms, the waves formation is discussed and some numerical results are reported. Finally in 5 equilibria with active and not active constraints for the maximization problem are discussed and the Bullwhip effect is analyzed in the case of nodes with more incoming sub-chains and one outgoing sub-chain.

2. Real supply networks.

In what follows, we want to give some examples of real supply networks and focus on some characteristics, that can be useful for a mathematical description of production processes. In particular, the aim is to describe rules according to which the goods are addressed from incoming sub-chains to the outgoing ones in order to motivate the introduction of the two algorithms described in Section 1.

Let us analyze a supply network for assembling wine bottles, described in Fig. 1 (left). Bottles coming from arc I_1 are sterilized in node 1. Then, the sterilized

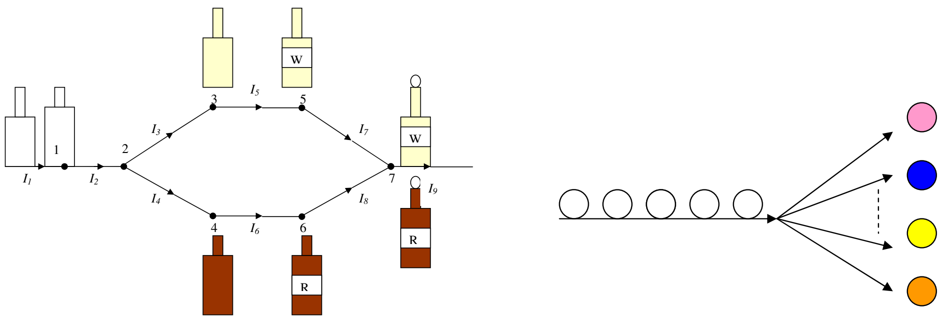

FIGURE 1. Wine production network (left) and balls production network (right).

bottles, with a certain probability α are directed to node 3, where white wine is bottled, and with probability $1-\alpha$ to node 4, where the red wine is bottled. In nodes 5 and 6, bottles are labelled, respectively, for red and white wine. Finally, in node 7, produced bottles are corked. Assume that red and white wine bottles are produced using two different bottle shapes. The bottles are addressed from arc I_2 to the outgoing sub-chains I_3 and I_4 in which they are filled up with white or red wine according to the bottle shape and thus according to the final destination: production of white or red wine bottles. In a model able to describe this situation, the dynamics at the node 2 is solved using the RA1 algorithm, in fact it is not possible the redirection of bottles in order to maximize the production on both incoming and outgoing sub-chains, since bottles with white and red wine have different shapes.
A supply network of beach balls production is considered in Fig. 1 (right). The white balls are addressed towards n sub-chains in which they are colored using different colors. Since the aim of the factory is to maximize the balls production independently from the colors, a mechanism is realized which addresses the balls on the outgoing sub-chains taking into account their loads in such way to maximize flux on both incoming and outgoing sub-chains. It follows that a model realized to capture the behavior of the described supply network is based on rule RA2.

Let us now analyze an existing supply network where both algorithms shows up naturally: the one for chips production of the San Carlo enterprise (see [21]). The productive processes follows various steps, that can be summarized in this way: when potatoes arrive at the enterprise, they are subjected to a goodness test. After this test, everything is ready for chips production, that starts with potatoes wash in drinking water. After washing potatoes, they are skinned off, rewashed and subjected to a qualification test. Then, they are cut in thin stripes by an automatic machine, and, finally, washed and dried by an air blow. At this point, potatoes are ready to be fried in vegetable oil for some minutes and, after this, the surplus oil is dripped. Potatoes are then salted by a dispenser, that nebulizes salt spreading it on potatoes. An opportune chooser is useful to select the best products. The final phase of the process is given by potatoes confection. A simplified vision of the supply chain network is in Fig. 2 (top). In phases 1, 5 and 10 a discrimination is made in production in order to distinguish good and bad products. In such sense, we can say that there is a statistical percentage

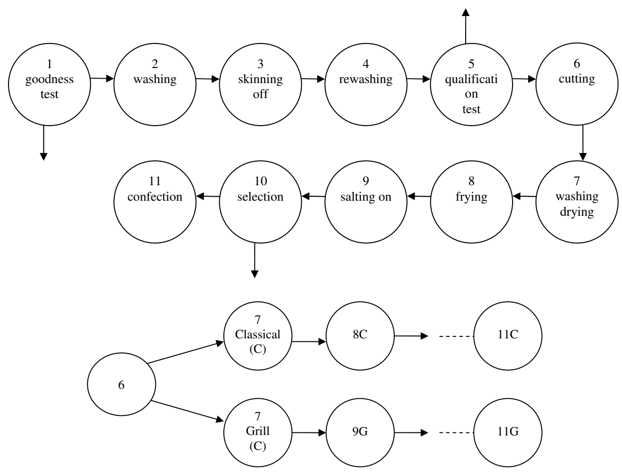

FIGURE 2. Graph of the supply network for chips production (top) and possible sub-chains (bottom).

α of product, that follows the production steps, while the percentage $1-\alpha$ is the product discarded (obviously, the percentage α can be different for different phases). Therefore, the goods routing in these nodes follows the algorithm RA1. On the other sied, phase 6 concerns the potatoes cut: as the enterprise produces different types of fried potatoes (classical, grill, light, stick, etc.), different ways of cutting potatoes must be considered. Assume that, for simplicity, there are only two types of potatoes production, then the supply network is as in Fig. Fig. 2 (bottom). If the aim is only the production maximization independently from the type, then the potatoes are addressed from node 6 towards the outgoing sub-chains according to the RA2 algorithm.

3. Basic definitions.

Let us consider a supply network consisting in $N+1$ sub-chains $I_1, \ldots, I_{N+1}$, modelled by real intervals $I_k = [a_k, b_k] \subset \mathbb{R}, k = 1, ..., N+1, a_k < b_k$, possibly with either $a_k = -\infty$ or $b_k = +\infty$ and M suppliers or processors $P_1, ..., P_M$ with certain throughput times and capacity. Each supplier processes a certain good, measured in units of parts. We assume that a node P consists of a processor, which decides how to manage the flow among sub-chains, with a maximal processing rate μ.

On each sub-chain I_k we consider the system

$$\left\{ \begin{array}{l} \rho_t + f_\varepsilon^k(\rho,\mu)_x = 0, \\ \mu_t - \mu_x = 0. \end{array} \right. \tag{3.1}$$

Each sub-chain I_k is thus characterized by a maximum density, a maximum rate and a flux f_ε^k. The flux is defined as in [9], therefore:

(F) $f_\varepsilon^k(\rho,\mu) = \left\{ \begin{array}{ll} \rho, & 0 \le \rho \le \mu, \\ \mu + \varepsilon(\rho-\mu), & \mu \le \rho \le \rho_k^{\max}, \end{array} \right.$

or alternatively

$$f_\varepsilon^k(\rho,\mu) = \left\{ \begin{array}{ll} \varepsilon\rho + (1-\varepsilon)\mu, & 0 \le \mu \le \rho, \\ \rho, & \rho \le \mu \le \mu_k^{\max}, \end{array} \right.$$

where $\rho_k^{\max}$ and $\mu_k^{\max}$ are the maximum density and processing rate. From now on, we assume that ε is fixed and, for simplicity, we drop the indices thus indicate the flux by $f(\rho, \mu)$.

Remark 3.1. It is possible to generalize all following definitions and results to the case of different fluxes $f_{\varepsilon_k}^k$ for each sub-chain I_k (also choosing ε dependent on k). In fact, all statements are in terms of values of fluxes at endpoints of the sub-chains, thus it is sufficient that the ranges of fluxes intersect. Moreover, we can consider different slopes m_k for each sub-chain I_k, considering the following flux

$$f_\varepsilon^k(\rho, \mu) = \begin{cases} m_k \rho, & 0 \le \rho \le \mu, \\ m_k \mu + \varepsilon(\rho - \mu), & \mu \le \rho \le \rho_k^{\max}, \end{cases}$$

where $m_k \ge 0$ represents the velocity of each processor and is given by:

$$m_k = \frac{L_k}{T_k},$$

with L_k and T_k, respectively, fixed length and processing time of processor k.

We assume that the sub-chains are connected by some junctions. Each junction J is given by a finite number of incoming sub-chains and a finite number of outgoing sub-chains, thus we identify J with $((i_1, ..., i_n), (j_1, ... j_m))$ where the first n-tuple indicates the set of incoming sub-chains and the second m-tuple indicates the set of outgoing sub-chains. Each sub-chain can be incoming sub-chain at most for one junction and outgoing at most for one junction. Hence the complete model is given by a couple $(\mathcal{I}, \mathcal{P})$, where $\mathcal{I} = \{I_k : k = 1, ..., N+1\}$ is the collection of sub-chains and $\mathcal{P}$ is the collection of junctions.

The supply network evolution is described by a finite set of functions ρ_k, μ_k defined on $[0, +\infty[\times I_k$. On each sub-chain I_k, we say that $U_k := (\rho_k, \mu_k) : [0, +\infty[\times I_k \mapsto \mathbb{R}$ is a weak solution to (3.1) if, for every C^∞-function $\varphi : [0, +\infty[\times I_k \mapsto \mathbb{R}^2$ with compact support in $]0, +\infty[\times]a_k, b_k[$,

$$\int_0^{+\infty} \int_{a_k}^{b_k} \left(U_k \cdot \frac{\partial \varphi}{\partial t} + f(U_k) \cdot \frac{\partial \varphi}{\partial x} \right) dx\, dt = 0,$$

where

$$f(U_k) = \begin{pmatrix} f(\rho_k, \mu_k) \\ -\mu_k \end{pmatrix},$$

is the flux function of the system (3.1). For the definition of entropic solution, we refer to [5].

For a scalar conservation law, a Riemann problem is a Cauchy problem for an initial data of Heavyside type, that is piecewise constant with only one discontinuity. The solutions are formed by rarefactions and shocks (see [5], [7]).

Analogously, we call Riemann problem for a junction the Cauchy problem corresponding to an initial data which is constant on each supply sub-chain.

Definition 3.2. A Riemann Solver (RS) for the junction P with n incoming sub-chains and m outgoing ones consists in a map that associates to a Riemann data $(\rho_0, \mu_0) = (\rho_{1,0}, \mu_{1,0}, ..., \rho_{n,0}, \mu_{n,0}, \rho_{n+1,0}, \mu_{n+1,0}, ..., \rho_{n+m,0}, \mu_{n+m,0})$ at P a vector $(\hat{\rho}_0, \hat{\mu}_0) = (\hat{\rho}_1, \hat{\mu}_1, ..., \hat{\rho}_n, \hat{\mu}_n, \hat{\rho}_{n+1}, \hat{\mu}_{n+1}, ..., \hat{\rho}_{n+m}, \hat{\mu}_{n+m})$ so that the solution is given by the waves $(\rho_{i,0}, \hat{\rho}_i)$ and $(\mu_{i,0}, \hat{\mu}_i)$ on the sub-chain $I_i, i = 1, ..., n$ and

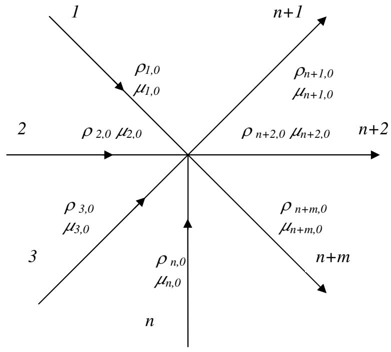

FIGURE 3. A junction.

by the waves $(\hat{\rho}_j, \rho_{j,0})$ on the sub-chain $I_j, j = n+1, ..., n+m$. We require the consistency condition

(CC) $RS(RS((\rho_0, \mu_0))) = RS((\rho_0, \mu_0))$.

Riemann Solvers, according to algorithms RA1 and RA2, will be defined in the next section.

Once a Riemann solver is assigned we can define admissible solutions at P.

Definition 3.3. Assume a Riemann Solver RS is assigned for the supplier P. Let $U = (U_1, ..., U_{n+m})$ be such that U is of bounded variation for every $t \geq 0$. Then U is an admissible weak solution of (3.1) related to RS at the junction P if and only if the following property holds for almost every t. Setting

$$\widetilde{U}_P(t) = (U_1(\cdot, b_1-), ..., U_n(\cdot, b_n-), U_{n+1}(\cdot, a_{n+1}+), ...U_{n+m}(\cdot, a_{n+m}+))$$

we have $RS(\widetilde{U}_P(t)) = \widetilde{U}_P(t)$.

Our aim is to solve the Cauchy problem on $[0, +\infty[$ for a given initial and boundary data as in next definition.

Definition 3.4. Given $\bar{U}_k : I_k \mapsto [0,1], k = 1, ..., N+1$, measurable BV functions, a collection of functions $U = (U_1, ..., U_{N+1})$, with $U_k : [0, +\infty[\times I_k \mapsto [0,1]$ continuous as functions from $[0, +\infty[$ into L^1_{loc} and $U_k(t, \cdot)$ BV function for almost every t, is an admissible solution to the Cauchy problem on the supply chain if U_k is a weak entropic solution to (3.1) on I_k, $U_k(0, x) = \bar{U}_k(x)$ a.e., and, at each supplier P_k, U is an admissible weak solution.

4. RIEMANN SOLVERS FOR SUPPLIERS.

In this Section we discuss Riemann Solvers, which conserve the flux at nodes. We consider two kinds of nodes:

- a node with more incoming sub-chains and one outgoing one;
- a node with one incoming sub-chain and more outgoing ones.

Let us fix a sub-chain I_k and analyze system (3.1): it is a system of conservation laws in the variables $U = (\rho, \mu)$:

$$U_t + F(U)_x = 0,$$

with flux function given by $F(U) = (f(\rho,\mu), -\mu)$.

The eigenvalues and eigenvectors are given by:

$$\lambda_1(\rho,\mu) \equiv -1, \qquad r_1(\rho,\mu) = \begin{cases} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & \text{if } \rho < \mu, \\ \\ \begin{pmatrix} -\frac{1-\varepsilon}{1+\varepsilon} \\ 1 \end{pmatrix}, & \text{if } \rho > \mu, \end{cases}$$

$$\lambda_2(\rho,\mu) = \begin{cases} 1 & \text{if } \rho < \mu, \\ \varepsilon & \text{if } \rho > \mu, \end{cases} \qquad r_2(\rho,\mu) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Hence the Hugoniot curves for the first family are vertical lines above the secant $\rho = \mu$ and lines with slope close to $-1/2$ below the same secant. The Hugoniot curves for the second family are just horizontal lines. Since we consider positive and bounded values for the variables, we fix the invariant region:

$$\mathcal{D} = \{(\rho,\mu) : 0 \le \rho \le \rho_{max},\ 0 \le \mu \le \mu_{max},$$
$$0 \le (1+\varepsilon)\rho + (1-\varepsilon)\mu \le (1+\varepsilon)\rho_{max} = 2(1-\varepsilon)\mu_{max}\},$$

see Fig. 4.

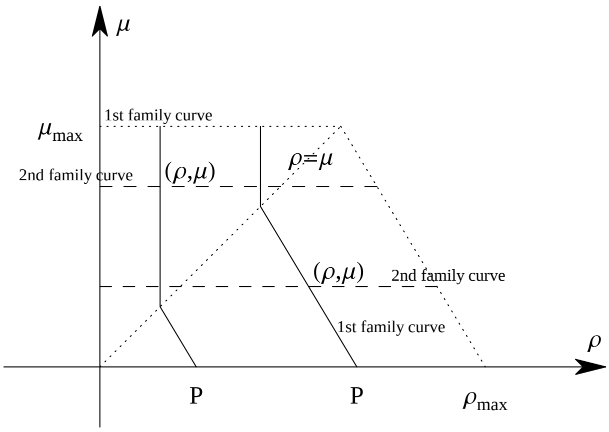

FIGURE 4. First and second family curves.

Observe that

$$\rho_{\max} = \mu_{\max}\frac{2}{1+\varepsilon}.$$

First we report some results proved in [9] for sequential supply chains.

Lemma 4.1. *Given an initial datum (ρ_0,μ_0), the maximum value of the density of the curve of the second family passing through (ρ_0,μ_0) and belonging to the invariant region is given by*

$$\rho^M(\mu_0) = \rho_{\max} - \mu_0\frac{\rho_{\max}-\mu_{\max}}{\mu_{\max}}. \tag{4.1}$$

Proposition 4.2. Given (ρ_0, μ_0), the minimal value of the flux at points of the curve of the first family passing through (ρ_0, μ_0) is given by:

$$f^{min}((\rho_0, \mu_0)) = \begin{cases} \frac{2\varepsilon}{1+\varepsilon}\rho_0, & \text{if } \rho_0 \le \mu_0, \\ \varepsilon\rho_0 + \frac{\varepsilon(1-\varepsilon)}{1+\varepsilon}\mu_0, & \text{if } \rho_0 > \mu_0. \end{cases}$$

We consider a node P with n incoming sub-chains and m outgoing ones and a Riemann initial datum $(\rho_{1,0}, \mu_{1,0}, ..., \rho_{n,0}, \mu_{n,0}, \rho_{n+1,0}, \mu_{n+1,0}, ..., \rho_{n+m,0}, \mu_{n+m,0})$.

The following Lemma holds:

Lemma 4.3. *On the incoming sub-chain, only waves of the first family may be produced, while on the outgoing sub-chain only waves of the second family may be produced.*

From Lemma 4.3, given the initial datum, for every Riemann Solver it follows that

$$\begin{array}{ll} \hat{\rho}_i = \varphi(\hat{\mu}_i), & i = 1, ..., n, \\ \hat{\mu}_j = \mu_{j,0}, & j = n+1, ..., n+m, \end{array} \tag{4.2}$$

where the function $\varphi(\cdot)$ describes the first family curve through $(\rho_{k,0}, \mu_{k,0})$ as function of $\hat{\mu}_k$. The expression of such curve changes at a particular value $\bar{\mu}_k$, given by:

$$\bar{\mu}_k = \begin{cases} \rho_{k,0}, & \text{if } \rho_{k,0} \le \mu_{k,0}, \\ \frac{1+\varepsilon}{2}\rho_{k,0} + \frac{1-\varepsilon}{2}\mu_{k,0}, & \text{if } \rho_{k,0} > \mu_{k,0}. \end{cases}$$

We define two different Riemann Solvers at a junction that represent two different routing algorithms:

RA1 We assume that

(A) the flow from incoming sub-chains is distributed on outgoing sub-chains according to fixed coefficients;

(B) respecting (A) the processor chooses to process goods in order to maximize fluxes (i.e., the number of goods which are processed).

RA2 We assume that the number of goods through the junction is maximized both over incoming and outgoing sub-chains.

For both routing algorithms we can maximize the flux of goods considering one of the two additional rules, introduced in [9]:

SC2 The objects are processed in order to maximize the flux with the minimal value of the processing rate.

SC3 The objects are processed in order to maximize the flux. If a solution with only waves in the density ρ exists, then such solution is taken, otherwise the minimal μ wave is produced.

To define Riemann problems according to rule RA1 and RA2 let us introduce the notation:

$$f_k = f(\rho_k, \mu_k).$$

Define the maximum flux that can be obtained by a wave solution on each production sub-chain:

$$f_k^{\max} = \begin{cases} \bar{\mu}_k, & k = 1, ..., n, \\ \mu_{k,0} + \varepsilon(\rho^M(\mu_{k,0}) - \mu_{k,0}), & k = n+1, ..., n+m. \end{cases}$$

Since $\hat{f}_i \in [f_i^{\min}, f_i^{\max} = \bar{\mu}_i], i = 1, ...n$ and $\hat{f}_j \in [0, f_j^{\max} = \mu_{j,0} + \varepsilon(\rho^M(\mu_{j,0}) - \mu_{j,0})], j = n+1, ..., n+m$ it follows that if

$$\sum_{i=1}^{n} f_i^{\min} > \sum_{j=n+1}^{n+m} f_j^{\max}$$

the Riemann Problem does not admit solution. Thus we get the following condition for the solvability of the supply network.

Lemma 4.4. *A necessary and sufficient condition for the solvability of the Riemann Problems is that*

$$\sum_{i=1}^{n} f_i^{\min} \leq \sum_{j=n+1}^{n+m} \mu_{j,0} + \varepsilon(\rho^M(\mu_{j,0}) - \mu_{j,0}).$$

Lemma 4.5. *A sufficient condition for the solvability of the Riemann Problems, independent of the initial data, is the following*

$$\sum_{i=1}^{n} \rho_i^{\max} \leq \sum_{j=n+1}^{n+m} \mu_j^{\max}.$$

Proof. Since $\hat{f}_i \in [f_i^{\min}, f_i^{\max}], i = 1, ...n$ and $\hat{f}_j \in [0, f_j^{\max}], j = n+1, ..., n+m$, the worst case to fulfill the condition of Lemma 4.4 happens when $f_i^{\min}$ assumes the greatest value and $f_j^{\max}$ the lowest one

$$\sum_{i=1}^{n} \varepsilon\rho_i^{\max} \leq \sum_{j=n+1}^{n+m} \varepsilon\mu_j^{\max}.$$

□

In what follows, first we consider a single junction $P \in \mathcal{P}$ with $n-1$ incoming arcs and 1 outgoing arc (shortly, a node of type $(n-1) \times 1$) and then a junction with 1 incoming arc and $m-1$ outgoing ones (shortly, a node of type $1 \times (m-1)$).

4.1. **One outgoing sub-chain.** In this case the two algorithms RA1 and RA2 coincide since there is only one outgoing sub-chain.

We fix a node P with $n-1$ incoming arcs and 1 outgoing one and a Riemann initial datum $(\rho_0, \mu_0) = (\rho_{1,0}, \mu_{1,0}, ..., \rho_{n-1,0}, \mu_{n-1,0}, \rho_{n,0}, \mu_{n,0})$. Let us denote with $(\hat{\rho}, \hat{\mu}) = (\hat{\rho}_1, \hat{\mu}_1, ..., \hat{\rho}_{n-1}, \hat{\mu}_{n-1}, \hat{\rho}_n, \hat{\mu}_n)$ the solution of the Riemann Problem. In order to solve the dynamics we have to introduce the priority parameters $(q_1, q_2, .., q_{n-1})$ which determine a *level of priority* at the junction of incoming sub-chains.

Let us define

$$\Gamma_{inc} = \sum_{i=1}^{n-1} f_i^{\max},$$
$$\Gamma_{out} = f_n^{\max},$$

and $\Gamma = \min\{\Gamma_{inc}, \Gamma_{out}\}$.

We analyze for simplicity the case in which $n = 3$, in this case we need only one priority parameter $q \in]0, 1[$. Think, for example, of a filling station for soda cans. The sub-chain 3 fills the cans, whereas sub-chains 1 and 2 produce plastic and aluminium cans, respectively.

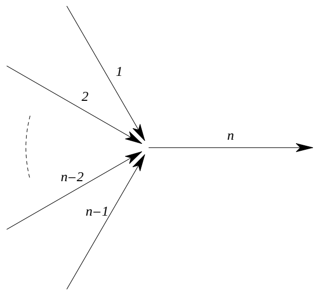

FIGURE 5. One outgoing sub-chain.

First, we compute $\hat{f}_i$ $i = 1, 2, 3$ and then $\hat{\rho}_i$ and $\hat{\mu}_i, i = 1, 2, 3$.

We have to distinguish two cases:

Case 1): $\Gamma = \Gamma_{inc}$.

Case 2): $\Gamma < \Gamma_{inc}$.

In the first case we set $\hat{f}_i = f_i^{\max}$, $i = 1, 2$. Let us analyze the second case in which we use the priority parameter q.

Not all objects can enter the junction, so let C be the amount of objects that can go through. Then qC objects come from first sub-chain and $(1 - q)C$ objects from the second. Consider the space (f_1, f_2) and define the following lines:

$$r_q : f_2 = \frac{1 - q}{q} f_1,$$

$$r_\Gamma : f_1 + f_2 = \Gamma.$$

Define P to be the point of intersection of the lines r_q and r_Γ. Recall that the final fluxes should belong to the region (see Fig. 6):

$$\Omega = \{(f_1, f_2) : 0 \leq f_i \leq f_i^{\max}, i = 1, 2\}.$$

We distinguish two cases:

a) P belongs to Ω,

b) P is outside Ω.

In the first case we set $\left(\hat{f}_1, \hat{f}_2\right) = P$, while in the second case we set $\left(\hat{f}_1, \hat{f}_2\right) = Q$, with $Q = proj_{\Omega \cap r_\Gamma}(P)$ where *proj* is the usual projection on a convex set, see Fig. 6.

Notice that $\hat{f}_3 = \Gamma$.

Remark 4.6. The reasoning can be repeated also in the case of $n - 1$ incoming sub-chains. In $\mathbb{R}^{n-1}$ the line r_q is given by $r_q = tv_q, t \in \mathbb{R}$, with $v_q \in \Delta_{n-2}$ where

$$\Delta_{n-2} = \left\{ (f_1, ..., f_{n-1}) : f_i \geq 0, i = 1, ..., n - 1, \sum_{i=1}^{n-1} f_i = 1 \right\}$$

is the $(n - 2)$ dimensional simplex and

$$H_\Gamma = \left\{ (f_1, ..., f_{n-1}) : \sum_{i=1}^{n-1} f_i = \Gamma \right\}$$

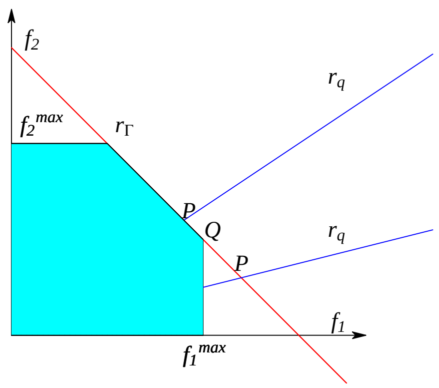

FIGURE 6. P belongs to Ω and P is outside Ω.

is a hyperplane. Since $v_q \in \Delta_{n-2}$, there exists a unique point $P = r_q \cap H_\Gamma$. If $P \in \Omega$, then we set $(\hat{f}_1, ..., \hat{f}_{n-1}) = P$. If $P \notin \Omega$, then we set $(\hat{f}_1, ..., \hat{f}_{n-1}) = Q = proj_{\Omega \cap H_\Gamma}(P)$, the projection over the subset $\Omega \cap H_\Gamma$. Observe that the projection is unique since $\Omega \cap H_\Gamma$ is a closed convex subset of H_Γ.

Let us compute $\hat{\rho}_k$ and $\hat{\mu}_k, k = 1, 2, 3$.

On the incoming sub-chains we have to distinguish two subcases:

Case 2.1): $\hat{f}_i = f_i^{\max}$. We set according to rules SC2 and SC3,

$$SC2: \begin{array}{l} \hat{\rho}_i = \bar{\mu}_i, \\ \hat{\mu}_i = \bar{\mu}_i, \end{array} \quad i = 1, 2,$$

$$SC3: \begin{array}{l} \hat{\rho}_i = \bar{\mu}_i, \\ \hat{\mu}_i = \max\{\bar{\mu}_i, \mu_{i,0}\}, \end{array} \quad i = 1, 2.$$

Case 2.2): $\hat{f}_i < f_i^{\max}$. In this case there exists a unique $\hat{\mu}_i$ such that $\hat{\mu}_i + \varepsilon(\varphi(\hat{\mu}_i) - \hat{\mu}_i) = \hat{f}_i$. According to (4.2), we set $\hat{\rho}_i = \varphi(\hat{\mu}_i), i = 1, 2$.

Observe that in case 2.1) $\hat{\rho}_i = \varphi(\hat{\mu}_i) = \bar{\mu}_i, i = 1, 2$.

On the outgoing sub-chain we have:

$$\hat{\mu}_3 = \mu_{3,0},$$

while $\hat{\rho}_3$ is the unique value such that $f_\varepsilon(\mu_{3,0}, \hat{\rho}_3) = \hat{f}_3$.

4.2. **One incoming sub-chain.** We fix a node P with 1 incoming arc and $m - 1$ outgoing ones and a Riemann initial datum $(\rho_0, \mu_0) = (\rho_{1,0}, \mu_{1,0}, \rho_{2,0}, \mu_{2,0}, ..., \rho_{m,0}, \mu_{m,0})$. Let us denote with $(\hat{\rho}, \hat{\mu}) = (\hat{\rho}_1, \hat{\mu}_1, \hat{\rho}_2, \hat{\mu}_2, ..., \hat{\rho}_m, \hat{\mu}_m)$ the solution of the Riemann Problem. Since we have more than one outgoing arc, we need to define the distribution of goods from the incoming arc.

Introduce the flux distribution parameters $\alpha_j, j = 2, ..., m$, where

$$0 < \alpha_j < 1, \ \sum_{j=2}^{m} \alpha_j = 1.$$

The coefficient α_j denotes the percentage of objects addressed from the arc 1 to the sub-chain j. The flux on the arc j is thus given by

$$f_j = \alpha_j f_1, \ j = 2, ..., m,$$

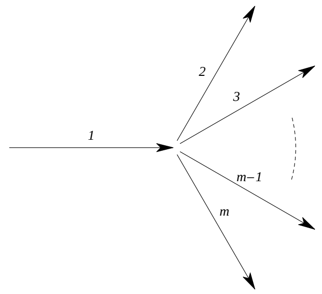

FIGURE 7. One incoming sub-chain.

where f_1 is the incoming flux on the arc 1.

Let us define

$$\Gamma_{inc} = f_1^{\max},$$
$$\Gamma_{out} = \sum_{j=2}^{m} f_j^{\max},$$

and $\Gamma = \min\{\Gamma_{inc}, \Gamma_{out}\}$.

We have to determine $\hat{\mu}_k$ and $\hat{\rho}_k, k = 1, ..., m$ for both algorithms RA1 and RA2.

4.2.1. *Riemann solver according to RA1.* Analyze the general case with m sub-chains. Consider, for example, the filling station for wine bottles of Section 2. The sub-chains I_3 and I_4 fill bottles with white and red wines, respectively, according to the bottle shapes. The dynamics at node 2 is solved using the algorithm we are going to describe. Since $\hat{f}_j \leq f_j^{\max}$ it follows that

$$\hat{f}_1 \leq \frac{f_j^{\max}}{\alpha_j}, j = 2, ..., m.$$

We set

$$\begin{array}{l} \hat{f}_1 = \min\{f_1^{\max}, \frac{f_j^{\max}}{\alpha_j}\}, \\ \hat{f}_j = \alpha_j \hat{f}_1, \end{array} \quad j = 2, ..., m.$$

On the incoming sub-chain we have to distinguish two subcases:

Case 1): $\hat{f}_1 = f_1^{\max}$. According to rule SC2 and SC3, respectively, we set

$$SC2: \begin{array}{l} \hat{\rho}_1 = \bar{\mu}_1, \\ \hat{\mu}_1 = \bar{\mu}_1, \end{array}$$

$$SC3: \begin{array}{l} \hat{\rho}_1 = \bar{\mu}_1, \\ \hat{\mu}_1 = \max\{\bar{\mu}_1, \mu_{1,0}\}. \end{array}$$

Case 2): $\hat{f}_1 < f_1^{\max}$. In this case there exists a unique $\hat{\mu}_1$ such that $\hat{\mu}_1 + \varepsilon(\varphi(\hat{\mu}_1) - \hat{\mu}_1) = \hat{f}_1$. According to (4.2), we set $\hat{\rho}_1 = \varphi(\hat{\mu}_1)$.

On the outgoing sub-chain we have:

$$\hat{\mu}_j = \mu_{j,0}, j = 2, 3,$$

while $\hat{\rho}_i$ is the unique value such that $f_\varepsilon(\mu_{j,0}, \hat{\rho}_j) = \hat{f}_j, j = 2, 3.$

4.2.2. *Riemann solver according to RA2.* Let us analyze for simplicity the case in which $m = 3$, in this case we need only one distribution parameter $\alpha \in]0, 1[$. Think, for example, the supply network of beach production described in Section 2. The dynamics at the node is solved according to the algorithm RA2. Compute $\hat{f}_k, k = 1, 2, 3$.

We have to distinguish two cases:

Case 1): $\Gamma = \Gamma_{out}$.

Case 2): $\Gamma < \Gamma_{out}$.

In the first case we set $\hat{f}_j = f_j^{\max}, j = 2, 3$. Let us analyze the second case in which we use the priority parameter α.

Not all objects can enter the junction, so let C be the amount of objects that can go through. Then αC objects come from the first sub-chain and $(1-\alpha)C$ objects from the second. Consider the space (f_2, f_3) and define the following lines:

$$r_\alpha : f_3 = \frac{1-\alpha}{\alpha} f_2,$$
$$r_\Gamma : f_2 + f_3 = \Gamma.$$

Define P to be the point of intersection of the lines r_α and r_Γ. Recall that the final fluxes should belong to the region:

$$\Omega = \left\{ (f_2, f_3) : 0 \le f_j \le f_j^{\max}, j = 2, 3 \right\}.$$

We distinguish two cases:

a) P belongs to Ω,

b) P is outside Ω.

In the first case we set $\left(\hat{f}_2, \hat{f}_3\right) = P$, while in the second case we set $\left(\hat{f}_2, \hat{f}_3\right) = Q$, with $Q = proj_{\Omega \cap r_\Gamma}(P)$ where $proj$ is the usual projection on a convex set. Observe that $\hat{f}_1 = \Gamma$.

Again, we can extend the reasoning to the case of $m-1$ outgoing sub-chains as for the incoming sub-chains defining the hyperplane

$$H_\Gamma = \left\{ (f_2, ..., f_m) : \sum_{j=2}^{m} f_j = \Gamma \right\}$$

and choosing a vector $v_\alpha \in \Delta_{m-2}$. Moreover, we compute $\hat{\rho}_k$ and $\hat{\mu}_k$ in the same way described for the Riemann Solver RA1.

Remark 4.7. An alternative way of choosing the vector v_α is the following. We assume that a traffic distribution matrix A is assigned, then we compute $\hat{f}_1$, and choose $v_\alpha \in \Delta_{m-2}$ by

$$v_\alpha = \Delta_{m-2} \cap \left\{ tA(\hat{f}_1) : t \in \mathbb{R} \right\}.$$

Remark 4.8. The classical Kruzkov entropy inequalities at nodes ([5]) read

$$\sum_{inc} sgn(\rho - k)(f(\rho) - f(k)) \ge \sum_{out} sgn(\rho - k)(f(\rho) - f(k))$$

where the sums are respectively over incoming and outgoing sub-chains and k is arbitrary. The fluxes are always monotone with respect to ρ, while the the precise values taken by fluxes and densities on the sub-chains may be different. Thus we can not expect the inequality to hold in general.

4.3. **Waves production.** Let us discuss now the waves production on an incoming sub-chain and on an outgoing one with initial datum $(\rho_{i,0}, \mu_{i,0})$ and $(\rho_{j,0}, \mu_{j,0})$, respectively. Since the load dynamic is described by a conservation law in ρ and an evolution equation in μ, we have $\rho-$waves and $\mu-$waves of two types: shocks waves which are discontinuities in ρ and/or μ travelling at a constant speed, and contact discontinuities, which separate two constant states with the same speed but different values. In particular, on an incoming sub-chain only waves of the first family can be produced. They are contact discontinuities in ρ and μ with speed $\lambda = -1$ connecting the states $\rho_{i,0}$ and $\hat{\rho}_i$ and $\mu_{i,0}$ and $\hat{\mu}_i$.

On the outgoing sub-chain only $\rho-$waves of the second family can be produced. Two cases must be considered:

Case a): $\rho_{j,0} \leq \mu_{j,0}$.
Case b): $\rho_{j,0} > \mu_{j,0}$.

In case a) we have to distinguish two subcases:

Case a1): If $\hat{\rho}_j \in [0, \mu_{j,0}]$ then the solution of the RP consists of a contact discontinuity connecting $\hat{\rho}_j$ and $\rho_{j,0}$ with speed 1 (for $t = 1$);

Case a2): If $\hat{\rho}_j \in]\mu_{j,0}, \mu_j^{\max}]$ then the solution of the RP consists of two shocks: one connecting $\hat{\rho}_j$ and $\mu_{j,0}$ with speed ε (for $t = 1$) followed by another shock connecting $\mu_{j,0}$ and $\rho_{j,0}$ travelling with speed 1 (for $t = 1$), see Fig. 8.

In case b) we have to consider two subcases:

Case b1): If $\hat{\rho}_j \in [0, \mu_{j,0}]$ then the solution of the RP consists of a shock wave connecting the states $\hat{\rho}_j$ and $\rho_{j,0}$ with speed (for $t = 1$) equal to the slope λ of the line connecting the two states

$$\lambda = \frac{\mu_{j,0} + \varepsilon(\rho_{j,0} - \mu_{j,0}) - \hat{\rho}_j}{\rho_{j,0} - \hat{\rho}_j}.$$

Case b2): If $\hat{\rho}_j \in], \mu_{j,0}, \mu_j^{\max}]$ then the solution of the RP consists of a contact discontinuity connecting $\hat{\rho}_j$ and $\rho_{j,0}$ with speed ε (for $t = 1$).

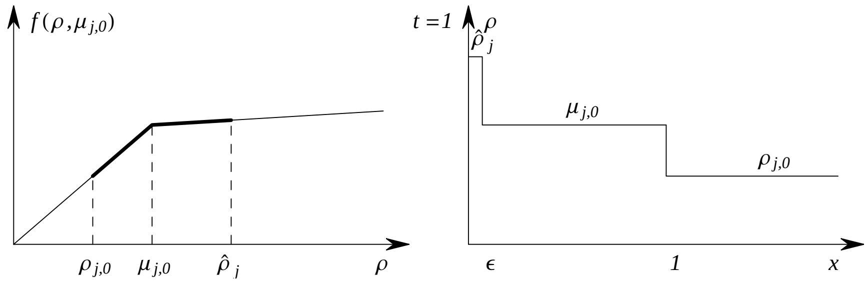

FIGURE 8. Waves production on an outgoing sub-chain: case a2).

In what follows we report the densities and production rates at the instant $t = 0$ and after some times (at $t = 1$) for different initial data using different routing algorithms. Since a constant state is an equilibrium for the single line model, a modification of the state may only appear initially at the junction. In Table 1 and in Fig. 9-10 we report the Riemann solver for a node of type 1×2 and assume $\varepsilon =$

$0.2, \mu_i^{\max} = 1, i = 1, 2, 3, \alpha = 0.8, (\rho_{1,0}, \rho_{2,0}, \rho_{3,0}) = (0.7, 0.1, 0), (\mu_{1,0}, \mu_{2,0}, \mu_{3,0}) = (1, 0.2, 1)$. Observe that the algorithm RA2 redirects the goods, in fact taking into account the initial loads of the outgoing sub-chains, the number of goods processed by the sub-chain with density $\rho_{3,0} = 0$ increases.

	RA1		RA2	
	SC2	SC3	SC2	SC3
$\hat{f}_i$	$(0.58, 0.47, 0.12)$	$(0.58, 0.47, 0.12)$	$(0.7, 0.47, 0.23)$	$(0.7, 0.47, 0.23)$
$\hat{\rho}_i$	$(0.82, 1.53, 0.12)$	$(0.82, 1.53, 0.12)$	$(0.7, 1.53, 0.23)$	$(0.7, 1.53, 0.23)$
$\hat{\mu}_i$	$(0.52, 0.2, 1)$	$(0.52, 0.2, 1)$	$(0.7, 0.2, 1)$	$(1, 0.2, 1)$

TABLE 1. A node of type 1×2.

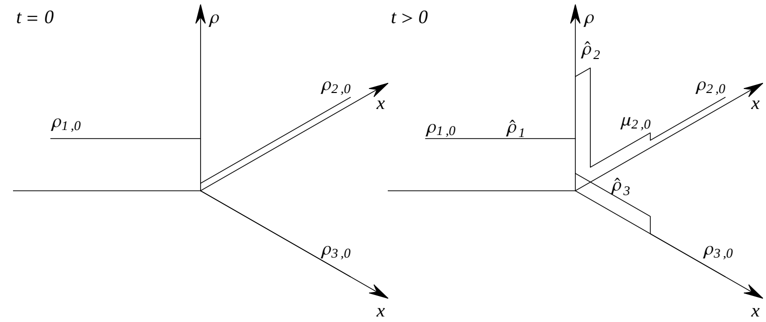

FIGURE 9. A RP for the RA2-SC3 algorithm: the initial density and the density after some times.

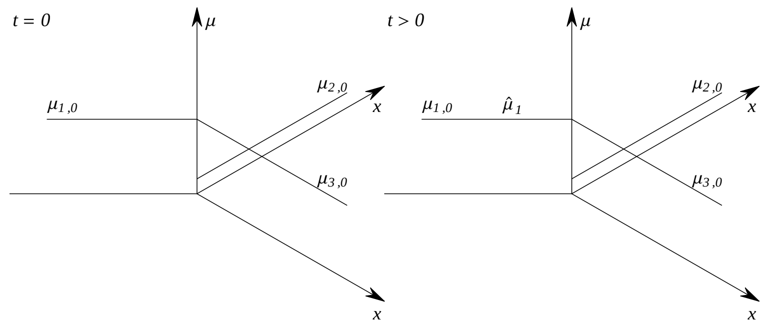

FIGURE 10. A RP for the RA2-SC3 algorithm: the initial production rate and the production rate after some times.

In Table 2 and in Fig. 11-12 we report numerical results for a node of type 2×1, and assume $\varepsilon = 0.2, \mu_i^{\max} = 1, i = 1, 2, 3, q = 0.6, (\rho_{1,0}, \rho_{2,0}, \rho_{3,0}) = (0.3, 0.7, 0.8)$, $(\mu_{1,0}, \mu_{2,0}, \mu_{3,0}) = (0.8, 0.7, 0.4)$.

	RA1=RA2	
	SC2	SC3
$\hat{f}_i$	$(0.3, 0.3, 0.6)$	$(0.3, 0.3, 0.6)$
$\hat{\rho}_i$	$(0.3, 1.1, 1.4)$	$(0.3, 1.1, 1.4)$
$\hat{\mu}_i$	$(0.3, 0.1, 0.4)$	$(0.8, 0.1, 0.4)$

TABLE 2. A node of type 2×1.

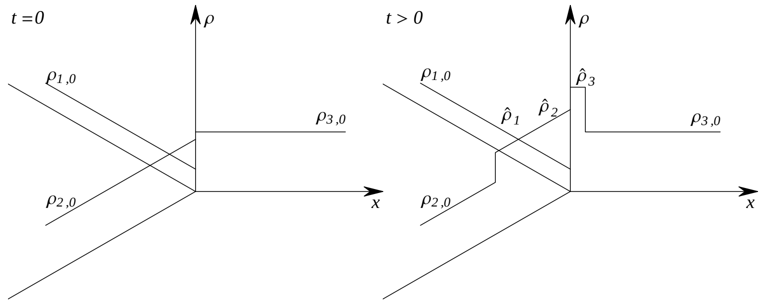

FIGURE 11. A RP for the SC2 algorithm: the initial density and the density after some times.

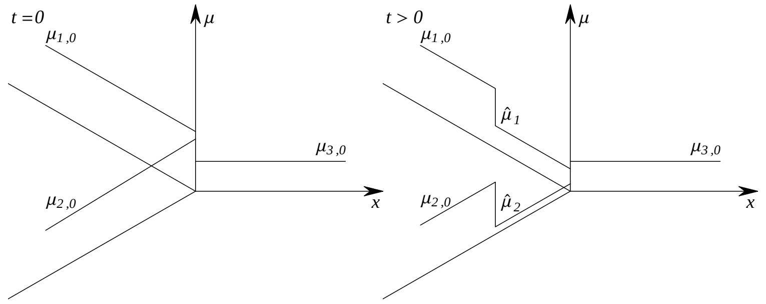

FIGURE 12. A RP for the SC2 algorithm: the initial production rate and the production rate after some times.

5. EQUILIBRIUM ANALYSIS.

In this Section we discuss the equilibria at nodes. We fix a node P and a Riemann initial datum (ρ_0, μ_0).

Definition 5.1. Define $(\hat{\rho}, \hat{\mu}) = RS((\rho_0, \mu_0))$. The datum (ρ_0, μ_0) is an equilibrium if

$$(\hat{\rho}, \hat{\mu}) = RS((\rho_0, \mu_0)) = (\rho_0, \mu_0).$$

We consider generic equilibria for the Riemann Problem at a junction. Let us distinguish two types of nodes, $(n-1) \times 1$ and $1 \times (m-1)$, and equilibria with active and not active constraints for the maximization problem.

5.1. **A node with one outgoing sub-chain.** If the n-th sub-chain is an active constraint then we have

$$\rho_n = \rho^M(\mu_n),$$

otherwise, if it is not an active constraint, we have:

$$\rho_n < \rho^M(\mu_n).$$

For the incoming sub-chains I_i, $i = 1, ..., n-1$, we have the following. If the i-th sub-chain is an active constraint then

$$\begin{array}{ll} SC2: & \mu_i = \rho_i \\ SC3: & \mu_i \geq \rho_i \end{array}, i = 1, ..., n-1.$$

Otherwise

$$\rho_i \geq \mu_i.$$

The equilibria are reported in Figure 13 and 14. In the latter the equilibria for the algorithm SC2 are depicted in bold, and those for the algorithm SC3 in bold and grey.

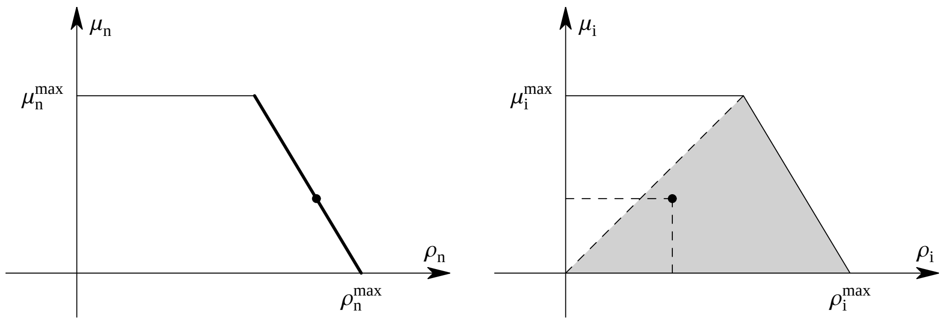

FIGURE 13. The outgoing sub-chain is an active constraint and the incoming ones are not active constraints.

The first type of equilibria (Figure 13) represents the situation in which the outgoing sub-chain exhibit the maximal production effort, while the incoming sub-chains adjust accordingly their production flows. In practice we expect this situation to show up frequently.
The second type of equilibria (Figure 14) represents the situation in which the incoming sub-chains have a low level of part densities and, consequently, the outgoing sub-chain is not used at maximal level. In other words, the whole plant is not used in an appropriate way and a re-building is in oder. Either the incoming sub-chains should be powered or the outgoing ones should be restricted. The first solution would improve the production rate, while the second would lower the production costs.

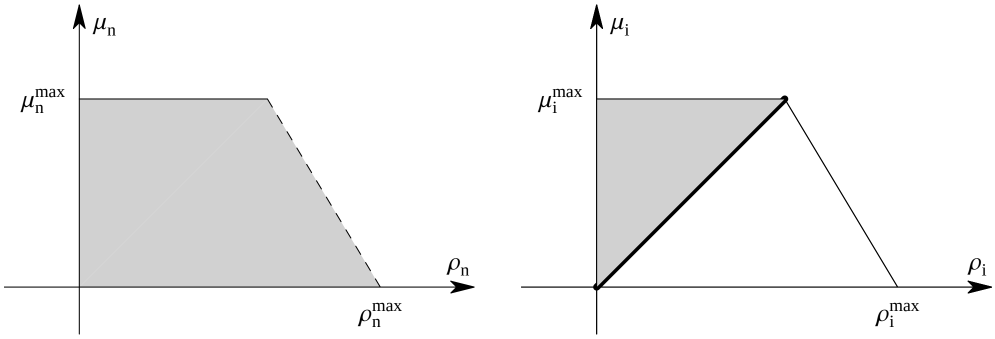

FIGURE 14. The incoming sub-chains are active constraints and the outgoing one is not an active constraint.

5.2. **A node with one incoming sub-chain.** The equilibria for the two algorithms RA1 and RA2 coincide. In particular, if the incoming sub-chain is an active constraint then

$$SC2: \quad \rho_1 = \mu_1,$$
$$SC3: \quad \rho_1 \leq \mu_1,$$

otherwise $\rho_1 \geq \mu_1$. For the outgoing sub-chains I_j, $j = 2, ..., m$, the following holds. If I_j is an active constraint then $\rho_j = \rho^M(\mu_j)$, for both SC2 and SC3 algorithms. Otherwise $\rho_j < \rho^M(\mu_j)$.
For both algorithms RA1 and RA2, the case of incoming sub-chain as active constraint should happen only with $\rho_1 = \mu_1$, in such a way that the goods fill up appropriately the sub-chain. Otherwise the incoming sub-chain should be powered. The situation for outgoing sub-chains as active constraints is different. In fact, the latter represents a projecting error for the algorithm RA1, while it may well happen for RA2.

5.3. **Bullwhip effect.** The Bullwhip effect is a well known oscillation phenomenon in supply chain theory, see [8]. Since the effect consists in oscillations moving backwards, we restrict ourselves to the most interesting case of nodes with $n-1$ incoming sub-chains and one outgoing sub-chain.

To study the Bullwhip effect, we compute the oscillations on incoming sub-chains produced by the interaction with the node of a wave from the outgoing one. Since the wave must have negative speed, it is a first family wave. To fix notation, let (ρ^-, μ^-) be an equilibrium configuration at the node and $((\rho_n^-, \mu_n^-), (\tilde{\rho}_n, \tilde{\mu}_n))$ the wave coming to the node. In general, we denote with $-$ and $+$ the values before and after the interaction, while by Δ we indicate the jump in the values from the left to the right along waves travelling on sub-chains.
The effect of the interaction of the wave is the production of $n-1$ waves on the incoming sub-chains.

The oscillation amplitude in the production rate before the interaction is given by:

$$\Delta\mu^- = \tilde{\mu}_n - \mu_n^-.$$

The maximum flux on the outgoing sub-chain as function of μ is the following

$$f_n^{\max}(\mu) = \mu \frac{1-\varepsilon}{1+\varepsilon} + \varepsilon \rho_n^{\max},$$

thus it is an increasing function. It follows that the oscillation of the flux after the interaction is

$$\Delta f^+ = \frac{1-\varepsilon}{1+\varepsilon} \Delta \mu^-.$$

Assume first that the incoming sub-chains are not active constraints. Then for both algorithms SC2 and SC3, we have $\rho_i^- \geq \mu_i^-, i = 1, ..., n-1$. Then the first family curve passing through $(\rho_{i,}^-, \mu_i^-)$, belonging to the region $\rho \geq \mu$, is given by

$$\rho = \rho_i^- + (\mu - \mu_i^-)(-\frac{1-\varepsilon}{1+\varepsilon}).$$

From which, for small oscillations we obtain

$$\Delta \rho^+ = -\frac{1-\varepsilon}{1+\varepsilon} \Delta \mu^+.$$

If the oscillation is not small the same relation holds with an inequality sign. Observe that

$$\Delta f^+ = \Delta \mu^+ (1-\varepsilon) + \varepsilon \Delta \rho^+ = \frac{1-\varepsilon}{1+\varepsilon} \Delta \mu^+,$$

from which

$$\Delta \mu^+ = \frac{1+\varepsilon}{1-\varepsilon} \Delta f^+,$$

and then

$$\Delta \mu^+ = \Delta \mu^-.$$

Assume now that the incoming sub-chains are active constraints. This means that $\mu_i^- = \rho_i^-$ for the SC2 algorithm and $\mu_i^- \geq \rho_i^-$ for the SC3 algorithm. Along the curve of the first family belonging to the region $\rho \leq \mu$ we have $\Delta f = 0$, i.e. a dumping effect is possible. On the contrary, in the region $\rho \geq \mu$ we have

$$\Delta f = \frac{1-\varepsilon}{1+\varepsilon} \Delta \mu.$$

Consider first the case of the SC2 algorithm. In case the first family wave from the outgoing road increases the flux, then it is reflected as a second family wave. In the opposite case, we get the same estimates as above.

Consider now the case of the SC3 algorithm. In case the first family wave from the outgoing road increases the flux, then it is again reflected as a second family wave. In the opposite case, we get:

$$\Delta \mu^+ = \Delta \mu^- + (\mu_i^- - \rho_i^-)$$

with an increase in the production rate oscillation.

Concluding we get the following:

Proposition 5.2. The algorithm SC3 may produce the Bullwhip effect. On the contrary, the algorithm SC2 conserves oscillations or produce a dumping effect, thus not permitting the Bullwhip effect.

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