On Non-Controllability of the Viscous Burgers Equation and Unbounded Entropy Solutions to Scalar Conservation Laws

Boris Andreianov

Institut Denis Poisson, University of Tours, France

We contribute a negative answer to the question raised by J.-M. Coron in [1] concerning controllability of non-zero constant states $C \neq 0$ for the viscous Burgers equation in a box:

$$u_t + \left(u^2/2\right)_x = u_{xx} \text{ in } (0,1) \times (0,T). \qquad \textbf{(BE)}$$

$$\underline{\textit{Question:}}\ \textit{Does there exist } u \in \mathrm{L}^2((0,1)\times(0,T)) \textit{ satisfying } (\mathbf{BE}) \textit{ such that for all } x \in (0,1), u(\cdot,0) = 0 \text{ and } u(\cdot,T) = C\ ? \qquad (\mathbf{Q})$$

Several positive answers, for couples (C,T) satisfying among other assumptions $|C|T > 1$, were obtained, e.g., via the Hopf-Cole reduction to the heat equation.

Based on the elementary observation that states with $C \neq 0$, $|C|T \leq 1$ are not attainable for the inviscid Burgers equation in the classical Kruzhkov entropy solution setting, we combine scaling and vanishing viscosity techniques to show that for (**BE**), the non-attainability persists for large $|C|$ and accordingly small T, under the additional "limited amplification" constraint $\|u\|_\infty \leq LC$ ($L \geq 1$ being fixed). More general data can be addressed with the same method.

To get closer to the setting of (**Q**), we develop a theory of L^2 ("unbounded") entropy solutions to the Burgers equation, in the Cauchy and Cauchy-Dirichlet setting; the above non-controllability result for (**BE**) is partly transferred to solutions satisfying the L^2 amplification assumption $\|u\|_2 \leq LT|C|$.

As a byproduct, we contribute an extension of classical Kruzhkov / Bardos-LeRoux-Nédélec theories to unbounded entropy solutions of critical integrability (ensuring that the flux belongs to L^1) for multidimensional scalar conservation laws, avoiding the technicalities of the general renormalization approach ([2]).

References

[1] J.M. Coron, Some open problems on the control of nonlinear partial differential equations. Perspectives in nonlinear partial differential equations. 215–243, Contemp. Math., 446, Amer. Math. Soc., Providence, RI, 2007.

[2] A. Porretta and J. Vovelle, L^1 solutions to first order hyperbolic equations in bounded domains, Comm. PDEs, 28 (2003), pp.381–408.

In collaboration with: S.S. Ghoshal, K. Koumatos