Control problems in models of the interaction between a population and individuals

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Motivation

The research is motivated by **crowd dynamics**, where the motions of a large number of people/animals/insects are studied.

Question: How to control such motions?

Here we consider situations where the population is influenced by a number of point objects (agents):

- a shepherd dog driving a herd of sheep;
- police officers controlling demonstrators;
- emergency crews driving a panicking crowd;
- predators separating preys from their herd;
- ...

Model

- The population is described by the set $X(t)$ it occupies.
- If there are no agents around, then the region occupied by the population spreads in all directions with a constant speed $c \geq 0$:

$$X(t) = B(X_0, ct)$$

- If there are k agents at the points $\xi_1, \dots, \xi_k$, then the population member located at x obtains an additional speed

$$v(x, \xi_1, \dots, \xi_k) = v(x, \xi)$$

Model

$$\left.\begin{array}{l} X_0 \subset \mathbb{R}^n \text{ is an initial set occupied by the population} \\ \xi = \xi(t) \text{ is a trajectory of the agents} \end{array}\right\} \Rightarrow$$

The set

$$X(t, X_0, \xi) \subset \mathbb{R}^n$$

occupied by the population at time t is the reachable set of

$$\begin{cases} \dot{x} \in v(x, \xi(t)) + B(0, c) \\ x(0) \in X_0 \end{cases} \tag{1}$$

The set-valued map $t \mapsto X(t, X_0, \xi)$ is a solution of (1) corresponding to the control ξ.

Control problem, where the state to be controlled is a set!

A. Bressan, D. Zhang, Set-Valued and Variational Analysis, 2012
(*2D case, 1 agent*)

Assumptions

(A1) The initial set X_0 is compact subset of $\mathbb{R}^n$.

(A2) The strategies $t \mapsto \xi(t) \subset \mathbb{R}^{kn}$ are locally Lipschitz continuous.

(A3) For every strategy ξ the set $\bigcup_{t \geq 0} \xi(t)$ is bounded.

(A4) The vector-field $v = v(x, \xi)$ is locally Lipschitz continuous and sublinear, i.e., there exists positive C such that

$$\big|v(x,\xi)\big| \leq C\big(1 + |x| + |\xi|\big) \qquad \forall x \in \mathbb{R}^n \;\; \forall \xi \in \mathbb{R}^{kn}.$$

Control problems

Confinement problem

Given $X_* \supseteq X_0$, find a strategy ξ such that $X(t, X_0, \xi) \subseteq X_*$ for all $t \geq 0$.

Steering problem

Given X_*, find a strategy ξ such that $X(t, X_0, \xi) \subseteq X_*$ at some $t > 0$.

Distributed strategies

- Agents are described by a probability measure μ on $\mathbb{R}^{kn}$.
- The solution $X = X(t, X_0, \mu)$ corresponding to the control μ is the reachable set of

$$\begin{cases} \dot{x} \in v_\mu(x) + B(0,c) \\ x(0) \in X_0 \end{cases} \tag{2}$$

where

$$v_\mu(x) = \int_{\mathbb{R}^n} v(x,\xi)\,\mathrm{d}\mu(\xi)$$

Checking confinement

A strategy μ is confining

$$\Updownarrow$$

The target set X_* is strongly invariant for (2)

$$\Updownarrow$$

$$\forall x \in \partial X_* \;\; \forall \mathbf{n} \in N^P_{X_*}(x) \qquad v_\mu(x) \cdot \mathbf{n} \leq -c$$

Example (spherically symmetric vector-field)

Consider the case of one agent

$$k = 1,$$

spherically symmetric vector filed

$$v(x, \xi) = \psi(|x - \xi|)(x - \xi),$$

and target set

$$X_* = B(0, R_*).$$

Example

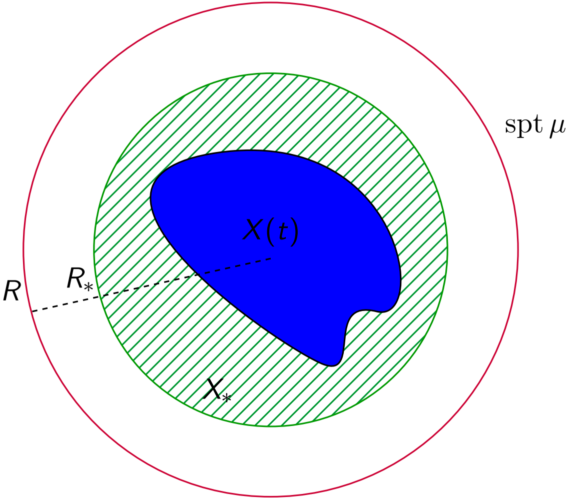
The distributed strategy

$$\mu(A) = \mathcal{H}^{n-1}\big(\partial B(0,R) \cap A\big)$$

is confining whenever

$$\frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^\pi \psi\left(\sqrt{R^2 + R_*^2 - 2R_*\, R\, \cos\vartheta}\right) (R_* - R\, \cos\vartheta)\, \sin^{n-2}\vartheta \,\mathrm{d}\vartheta \leq -c$$

Example



Approximation

Proposition

Let μ be a probability measure having a compact support. Then, for every $\varepsilon > 0$ there exists a locally Lipschitz continuous $\xi = \xi(t)$ such that

$$d_H\big(X(t, X_0, \xi), X(t, X_0, \mu)\big) < \varepsilon \qquad \forall t > 0$$

where d_H is the Hausdorff distance.

Example

In 2D case the Lipschitz continuous strategy

$$\xi_\varepsilon(t) = \left(R \cos \frac{2\pi t}{\varepsilon},\ R \sin \frac{2\pi t}{\varepsilon} \right)$$

approximates the distributed strategy

$$\mu(A) = \frac{1}{2\pi R} \mathcal{H}^1(\partial B(0, R) \cap A)$$

as $\varepsilon \to 0$

Infinite time confinement

Theorem

Let

- the vector-field be defined by $v(x,\xi) = \psi(|x-\xi|)(x-\xi)$
- there exist positive R, R_*^-, R_*^+ with $R_*^- < R_*^+$ and such that for all $R_* \in [R_*^-, R_*^+]$

$$\frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^\pi \psi\left(\sqrt{R^2 + R_*^2 - 2R_*\, R\, \cos\vartheta}\right) (R_* - R\,\cos\vartheta)\,\sin^{n-2}\vartheta \,\mathrm{d}\vartheta < -c$$

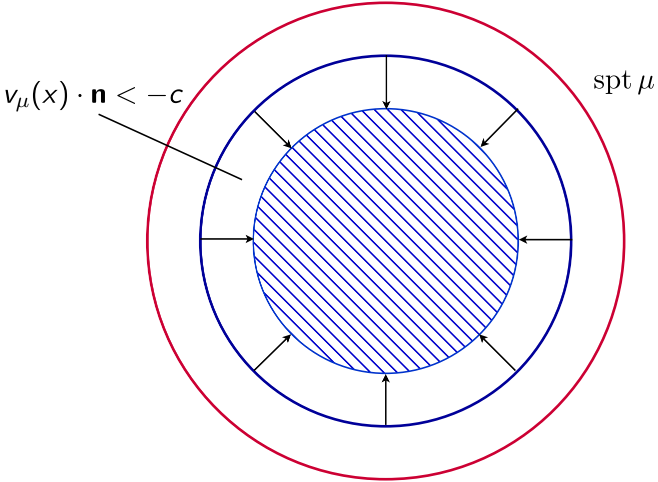
Then, there exists $\xi\colon \mathbb{R}^+ \to \partial B(0,R)$ such that

$$X_0 \subseteq B(0,R_*^-) \quad \Rightarrow \quad X(t,X_0,\xi) \subseteq B(0,R_*^+) \quad \forall t > 0$$

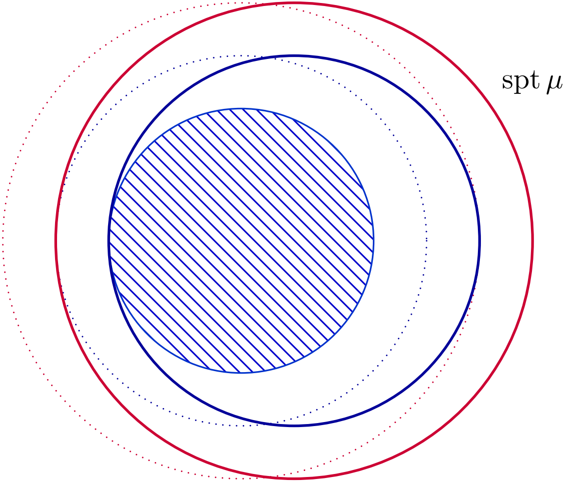
Confining

Too slow to confine

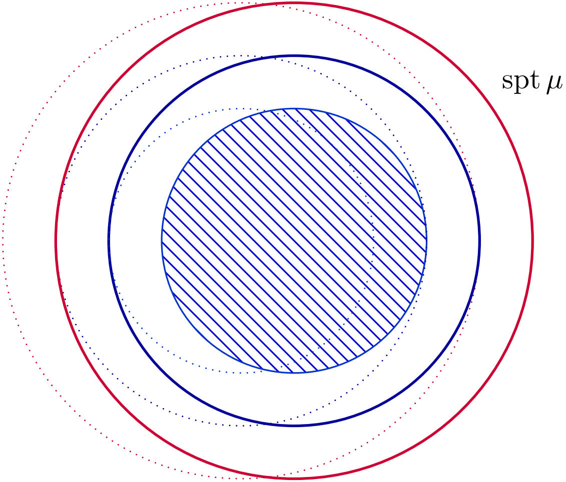
Steering



Steering



Steering



Steering

PDE-based model

- The population is described by a density function $\rho = \rho(t, x)$.
- Every population's member moves along a vector-field $v = v(t, x, \rho)$.

Then, ρ evolves according to the law of mass conservation

$$\begin{cases} \partial_t \rho + \operatorname{div}\left(\rho v(t, x, \rho)\right) = 0 \\ \rho(0) = \rho_0 \end{cases}$$

By solution we mean the usual entropy admissible solution.

Connection between the models

Assume that

- $\rho_0 = \rho_0(x)$ is a non-negative function with compact support,
- $\rho = \rho(t, x)$ and $X = X(t)$ are the solutions of

$$\begin{cases} \partial_t \rho + \operatorname{div}\left(\rho v(t,x)\right) = 0 \\ \rho(0,x) = \rho_0(x) \end{cases} \qquad \begin{cases} \dot{x} = v(t,x) \\ x(0) \in \operatorname{spt} \rho_0 \end{cases}$$

Method of characteristics $\quad \Rightarrow \quad \operatorname{spt} \rho(t, \cdot) = X(t)$

Connection between the models

Assume that

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$$\begin{cases} \partial_t \rho + \mathrm{div}\,\big(\rho v(\rho)\big) = 0 \\ \rho(0, x) = \rho_0(x) \end{cases} \qquad \begin{cases} \dot{x} \in B(0, c) \\ x(0) \in \mathrm{spt}\,\rho_0 \end{cases}$$

with $c = \max\left\{ |v(r) + r v'(r)| \; : \; r \in [0, \max \rho_0] \right\}$

Kružkov's Theorem $\Rightarrow$ $\mathrm{spt}\,\rho(t, \cdot) \subseteq X(t)$

Since in this case $\rho(t, \cdot) \in \mathbf{L}^1(\mathbb{R}^n)$, by $\mathrm{spt}\,\rho(t, \cdot)$ we mean the support of the measure $\rho(t, \cdot)\mathcal{L}^n$.

Connection between the models

Theorem

- $\rho_0 = \rho_0(x)$ be a non-negative function with compact support;
- $\rho = \rho(t,x)$ be an entropy admissible solution of

$$\begin{cases} \partial_t \rho + \mathrm{div}\,\big(\rho v(t,x) + \rho G(\rho)\big) = 0 \\ \rho(0,x) = \rho_0(x) \end{cases}$$

  satisfying the estimate $0 \le \rho(t,x) \le M$ for all t, x;
- $X = X(t)$ be a solution of

$$\begin{cases} \dot{x} \in v(t,x) + B(0,c) \\ x(0) \in \mathrm{spt}\,\rho_0 \end{cases}$$

  with $c = \max\big\{|G(r) + rG'(r)| \;:\; r \in [0,M]\big\}$.

Then

$$\mathrm{spt}\,\rho(t,\cdot) \subseteq X(t) \qquad \forall t \ge 0.$$

Control problems for a PDE

Control system:

$$\begin{cases} \partial_t \rho + \mathrm{div}\left(\rho v(x, \xi(t)) + \rho G(\rho)\right) = 0 \\ \rho(0, x) = \rho_0(x) \end{cases} \tag{3}$$

Here $\xi = \xi(t)$ is a Lipschitz continuous control.

Confinement problem

Given $X_* \supseteq \mathrm{spt}\, \rho_0$, find a strategy ξ such that $\mathrm{spt}\, \rho(t, \cdot) \subseteq X_*$ for all $t \geq 0$.

Steering problem

Given X_*, find a strategy ξ such that $\mathrm{spt}\, \rho(t, \cdot) \subseteq X_*$ at some $t > 0$.

Connection between the models

Corollary

Let $r \mapsto rG(r)$ be globally Lipschitz continuous on $[0, \infty[$.
Then

$$\xi \text{ is a confining (steering) strategy for (1)}$$
$$\Downarrow$$
$$\xi \text{ is a confining (steering) strategy for (3)}$$